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COMPUTER OBSERVATIONS OF CYCLES IN CUBIC GRAPHS

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Abstract—We investigate different instances where the available computer-generated empirical evidence concerning certain properties of cycles in cubic graphs either appears to clash with theory or, at best, is counterintuitive. Perhaps the most interesting of these involves a conjecture readily suggested by the data but for which a counterexample has been proved to exist. However, no such counterexample has yet been found.

1. INTRODUCTION

The compilation by Bussemaker *et al.* [1] of all cubic graphs of order n, $4 \leqslant n \leqslant 14$, suggests varied problems and conjectures. We discuss three problems and attendant conjectures involving the number of cycles in cubic graphs. In two of these the empirical evidence provided by the compilation appears, at first glance, counterintuitive and in the remaining the empirical evidence readily suggests a conjecture which we prove to be false.

Since cubic graphs necessarily have even order, it is understood throughout, that when we speak of cubic graphs of order n, n is even. Finally, the circled numbers associated with the graphs pictured in Figs 1 and 3 are the graph identification numbers used in Ref. [1].

2. CUBIC GRAPHS WITH THE MAXIMUM NUMBER OF CYCLES

Let $f(n)$ be the maximum number of cycles possible in a cubic graph of order n. Entringer and Slater [2] showed that

$$2^{n/2} < f(n) < 2^{n/2+1}. \tag{1}$$

It was easily, but tediously, verified from the tables of Ref. [1] that when $4 \leqslant n \leqslant 14$, a cubic graph of order n has $f(n)$ cycles if it has maximum girth and we conjectured that this held in general.† It may, at first, seem counterintuitive that we sacrifice small cycles to achieve a large total number of cycles in a graph. However, it is easily shown that a cubic graph of order n has girth at most $2 \log n/\log 2$. But, as demonstrated by Clark and the present authors in Ref. [3], such a graph has at most

$$n2^{2 \log n/\log 2} = n^3 \tag{2}$$

cycles with size at most the maximum girth possible. Consequently, the contribution of small cycles toward $f(n)$ is insignificant.

Entringer and Slater [2, 4] conjectured that, with the exception of K_4 and $K_{3,3}$, every cubic graph G contains an edge that lies in at most half the cycles of G. This conjecture, if correct, would allow the upper bound in expression (1) to be improved to $1.66 \cdot 2^{n/2}$. In Ref [2] a stronger conjecture was formulated, a special case of which is the following.

Conjecture

If G is any cubic graph other than K_4 and $K_{3,3}$ then every node of G is incident with an edge that lies in at most half the cycles of G.

† This has been disproved by David Guichard.

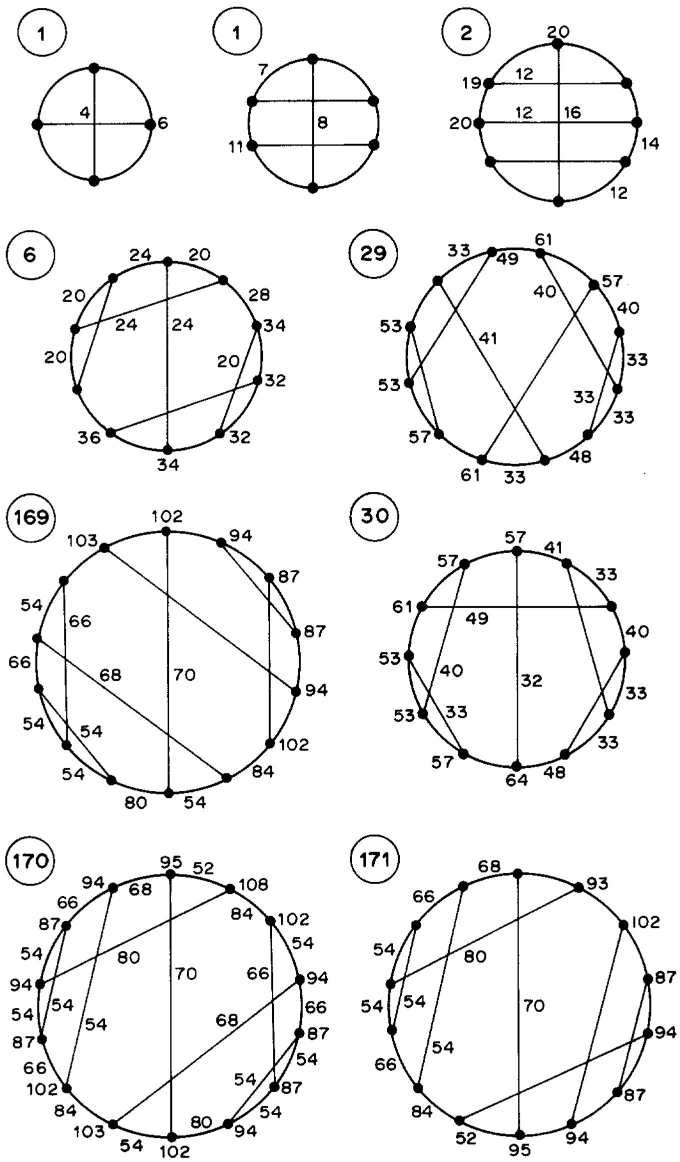

Fig. 1. Small 3-connected cubic graphs with the minimum number of cycles.

In Fig. 1 we picture certain cubic graphs for purposes to be discussed in the next section. In each of these, some of the edges are labeled with the number of cycles in which they lie and some of the nodes are similarly labeled. The labels for the remaining edges and nodes can be determined, in all cases, from the symmetries of the graphs and existing labels. The above conjecture is easily verified for these graphs by using Table 1.

3. CUBIC GRAPHS WITH THE MINIMUM NUMBER OF CYCLES

Clark *et al.* [3] characterized cubic graphs with connectivity 1 with the minimum number of cycles. The same was done for cubic graphs with connectivity 2. For 3-connected cubic graphs only widely disparate bounds were obtained for the minimum number of cycles possible in such graphs. These latter graphs are pictured in Fig. 1 for order n, $4 \leqslant n \leqslant 14$.

Table 1. The number of cycles in the graphs of Fig. 1

Graph identification number	1	1	2	6	29	30	169	170	171
Total number of cycles	7	14	26	46	79	79	133	133	133

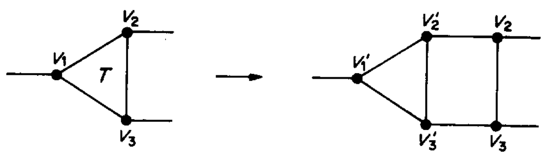

Fig. 2. Expanding a node to a triangle.

We now describe a recursive construction which generates these graphs.

Let $\mathscr{G}_4 = \{K_4\}$ and let $\mathscr{G}_{2m}, m \geqslant 3$, be the class of graphs G' obtained from the graphs $G \in \mathscr{G}_{2m-2}$ by choosing, from those nodes of G that lie in triangles, a node v that lies in the smallest number of cycles of G and expanding v to a triangle (see Fig. 2).

Observation

For $2 \leqslant m \leqslant 7$, the class $\mathscr{G}_{2m}$ consists of precisely those 3-connected cubic graphs of order $n = 2m$ that have the minimum number of cycles.

We now determine the number of cycles in these graphs. In this result F_m is the mth Fibonacci number, indexed so that $F_0 = 0$.

Theorem

If $G \in \mathscr{G}_{2m}, m \geqslant 2$, then:

(i) G has exactly two triangles, T_1 and T_2, when $m \geqslant 3$;
(ii) each triangle T of G has two nodes that lie in $F_{m+4} - 2$ cycles of G and one node that lies in $F_{m+4} + F_{m-1} - 3$ cycles of G;
(iii) G has $F_{m+5} - m - 4$ cycles.

Proof. Since $\mathscr{G}_6$ has only one member (graph number 1 of Fig. 1) and it has just two triangles, (i) follows by induction on m.

In proving (ii) we first note that every node of K_4 lies in $F_6 - 2 = F_6 + F_1 - 3 = 6$ cycles.

Suppose (ii) holds for some m and let T be a triangle of some member G of $\mathscr{G}_{2m}$. Throughout the remainder of the proof we denote by $\Psi(H; v)$ $(\Psi(H; uv))$ the number of cycles of the graph H containing the node v (edge uv, respectively) of H.

We label the nodes of T so that $\Psi(G; v_i) = F_{m+4} - 2$ for $i = 1, 2$ and $\Psi(G; u_3) = F_{m+4} + F_{m-1} - 3$. Let G' be the member of $\mathscr{G}_{2m+2}$ obtained from G be expanding v_1 to a triangle T' with nodes v_1', v_2' and v_3' as indicated in Fig. 2.

Let x_i, $i = 1, 2, 3$, be the number of cycles of G containing v_{i+1} and v_{i+2} but not v_i (indices are read mod 3); then

$$\Psi(G; v_i) = 1 + 2(x_1 + x_2 + x_3) - x_i \tag{3}$$

so that

$$x_i = \tfrac{2}{5}[\Psi(G; v_1) + \Psi(G; v_2) + \Psi(G; v_3)] - \Psi(G; v_i) - \tfrac{1}{5}. \tag{4}$$

Therefore,

$$\begin{aligned}\Psi(G'; v_1') &= 2 + 4(x_1 + x_2 + x_3) - 3x_1 \\ &= \tfrac{1}{5} - \tfrac{2}{5}[\Psi(G; v_1) + \Psi(G; v_2) + \Psi(G; v_s)] + 3\Psi(G; v_1).\end{aligned} \tag{5}$$

Consequently,

$$\Psi(G'; v_1') = \tfrac{1}{5} - \tfrac{2}{5}[3F_{m+4} + F_{m-1} - 7] + 3F_{m+4} - 6 = F_{m+5} + F_m - 3. \tag{6}$$

Also,

$$\begin{aligned}\Psi(G'; v_2') = \Psi(G'; v_3') &= 3 + 2x_1 + 3(x_2 + x_3) = \tfrac{7}{5} + \tfrac{6}{5}\Psi(G; v_1) \\ &\quad + \tfrac{1}{5}[\Psi(G; v_2) + \Psi(G; v_3)] = \tfrac{7}{5} + \tfrac{6}{5}[F_{m+4} - 2] + \tfrac{1}{5}[2F_{m+4} + F_{m-1} - 5] \\ &= F_{m+5} - 2.\end{aligned} \tag{7}$$

So that the proof of (ii) is complete by induction on m.

To prove (iii), we simply note that if $\Psi(H)$ denotes the number of cycles of a graph H then, assuming $\Psi(G) = F_{m+5} - m - 4$,

$$\begin{aligned}\Psi(G') &= \Psi(G) + \Psi(G'; v_2' v_3') = F_{m+5} - m - 4 + 2 + x_1 + 2(x_2 + x_3)\\ &= F_{m+5} - m - 2 + \Psi(G; v_1) - 1 = F_{m+5} - m - 3 + F_{m+4} - 2\\ &= F_{m+6} - (m+1) - 4. \qquad (8)\end{aligned}$$

Consequently, since $\Psi(K_4) = 7 = F_7 - 2 - 4$, (iii) follows by induction on m. □

The following now follows from the Observation and Theorem.

Fact

The 3-connected cubic graphs of order n, $4 \leqslant n \leqslant 14$, with the least number of cycles have exactly $F_{n/2+5} - n/2 - 4$ cycles.

We now show that there are arbitrarily large values of n for which the above statement is not true.

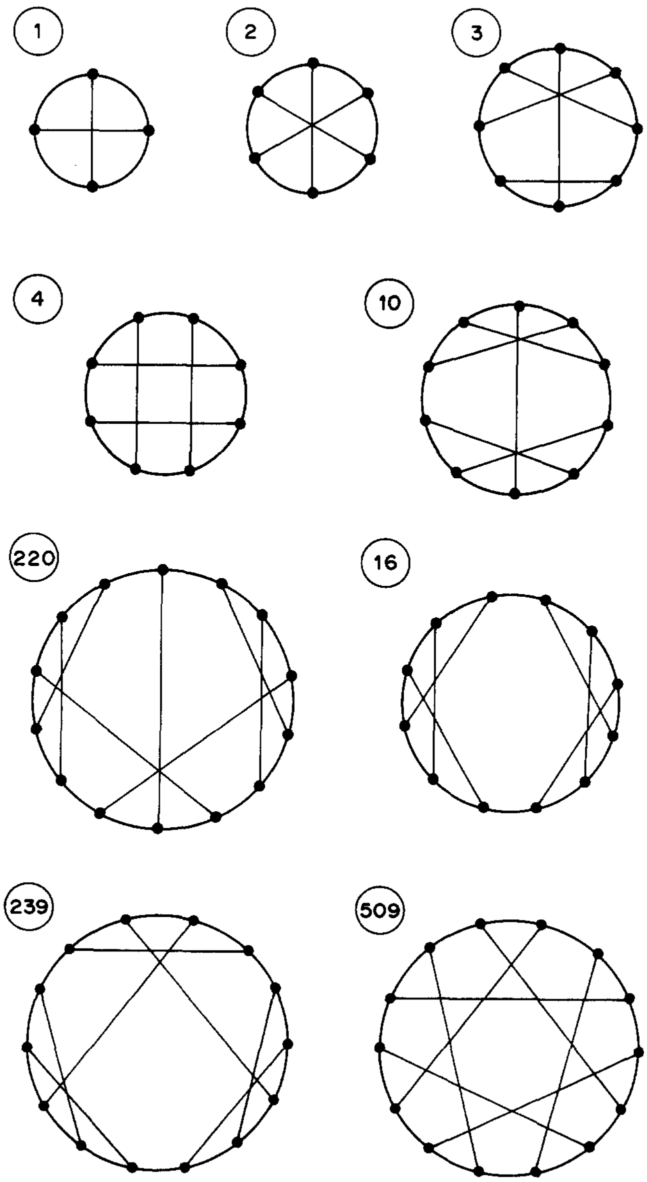

Fig. 3. Cubic graphs with the maximum number of Hamiltonian cycles.

Table 2. The number of Hamiltonian cycles in the graphs of Fig. 3

n	$f(n)$	Graph identification number
4	3	1
6	6	2
8	6	3, 4
10	12	10
12	16	16
14	24	220, 239, 509

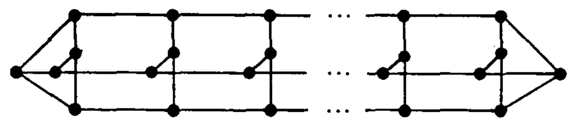

Fig. 4. Graphs with $3.2^{(n-2)/4}$ Hamiltonian cycles.

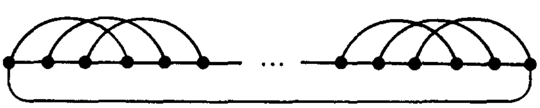

Fig. 5. Graphs with $2^{n/3}$ Hamiltonian cycles.

Bondy and Simonovits [5] constructed, recursively from the Petersen graph, 3-connected cubic graphs of order $n = 9^r \cdot 10$ in which the largest cycle has $8^r \cdot 9$ nodes, $r \geqslant 0$. Since, from an argument of Ref. [3], such graphs can have at most

$$n2^{9.8^r} = n \cdot 2^{9.8^{\log(n/10)/\log 9}} \leqslant n(2.026)^{n^{0.947}} \tag{9}$$

cycles and because

$$n(2.026)^{n^{0.947}} < 4.95\left(\frac{1+\sqrt{5}}{2}\right)^{n/2} - \frac{n}{2} - 5 < F_{n/2+5} - \frac{n}{2} - 4 \tag{10}$$

for $n \geqslant 10^{8.9}$, the pattern described in the Fact must eventually be broken.

4. CUBIC GRAPHS WITH THE MAXIMUM NUMBER OF HAMILTONIAN CYCLES

In Fig. 3 those cubic graphs of order n, $4 \leqslant n \leqslant 14$, having the maximum number, $f(n)$, of Hamiltonian cycles are pictured. The number of such cycles in each of these graphs is given in Table 2.

Two patterns emerge. The graphs with identification numbers 2, 10 and 220 are special cases of the family of 3-connected cubic graphs pictured in Fig. 4. It is easily verified that these graphs have $3 \cdot 2^{(n-2)/4}$ Hamiltonian cycles.

The graphs with identification numbers 2 and 16 are special cases of the family of graphs pictured in Fig. 5. When $n > 6$ these graphs obviously have connectivity 2 and, as is easily verified, have $2^{n/3}$ Hamiltonian cycles.

In general, increasing the connectivity of graphs appears to increase the total number of cycles they contain. Perhaps this will not be so when counting only Hamiltonian cycles.

5. CONCLUSION

In Section 2 we saw that empirical evidence suggests that the cubic graphs with the maximum total number of cycles are those with largest possible girth. Allender [6] has studied the related problem for digraphs. Specifically, he asks for the maximum number of cycles possible in a digraph of order n and girth $f(n)$ where $f(n)$ is a prescribed nonnegative function on the positive integers.

In Section 3 we showed that among the 3-connected cubic graphs with order n less than one billion, there were some with fewer than $F_{n/2+5} - n/2 - 4$ cycles. The problem of finding the smallest such graph remains to be resolved.

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